BAziz scientific research journal

Periodic ground states for the $\lambda-$model on the Cayley tree of order $k\geq 3$

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Abstract— In this paper I consider the $\lambda-$model on the Cayley tree of order $k\geq 3$. Assume that spin takes its values in the set $\{1,2,3\}$. I calculate energy of all unit balls and colletions assume of minimum value energies of unit balls and I describe periodic ground states for the considered model.

Index Terms—Cayley tree, $\lambda-$model, configuration, group, quotient group, periodic configuration, ground state, periodic ground state.

1 Introduction

The phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, see [1, 5]. The problem naturally arises on description of periodic ground states[1-11].

For the $\lambda-$model on the Cayley tree of order two, periodic ground states were studied in [2].

In this paper I consider periodic ground states for the $\lambda-$model on the Cayley tree of order $k\geq 3$.

2 Preliminaries

Detailed submission guidelines can be found on the author resources Let $\tau^k=(V,L), k\geq 1$ be a Cayley tree of order k, i.e., an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of τ^k. Let G_k denote the free product of $k+1$ cyclic groups $\{e,a_i\}$ of order 2 with generators $a_1,a_2,...,a_{k+1}$, i.e., let $a_i^2=e$. There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order k and the group G_k, see [1].

For each $x\in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x)=\{y\in G_k :< x,y >\in L\}$. The set $S_1(x)\setminus S(x)$ is a singleton. Let $x_{\downarrow}$ denote the (unique) element of this set.

Assume that spin takes its values in the set $\Phi=\{1,2,...,q\}$, $q\geq 2$. By a configuration σ on V we mean a function taking $\sigma: x\in V\to\sigma(x)\in\Phi$. The set of all configurations coincides with the set $\Omega=\Phi^V$.

Consider the quotient group $G_k/G_k^*=\{H_1,...,H_r\}$, where G_k^* is a normal subgroup of index r with $r\geq 1$.

Definition 2.1. *A configuration $\sigma(x)$ is said to be G_k^*-periodic if $\sigma(x)=\sigma_i$ for all $x\in G_k$ with $x\in H_i$. A*

$G_k -$ *periodic configuration is said to be translation invariant.*

By *period* of a periodic configuration we mean the index of the corresponding normal subgroup.

The Hamiltonian of the $\lambda -$ model [2] has a form,

$$H(\sigma) = \sum_{\langle x,y\rangle \in L} \lambda(\sigma(x),\sigma(y)) \tag{1}$$

In what follows, we restrict ourself to the case $\Phi = \{1,2,3\}$ and $k \geq 3$, and for the sake simplicity, we consider the following function:

$$\lambda(i,j) = \begin{cases} \bar{a}, \ if \ |i-j| = 2, \\ \bar{b}, \ if \ |i-j| = 1, \\ \bar{c}, \ if \ i = j, \end{cases} \tag{2}$$

where $\bar{a}, \bar{b}, \bar{c} \in R$.

3 Ground states

Let M be the set of unit balls with vertices in V, i.e. $M = \{x, S_1(x), \forall x \in V\}$. We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b.

We define the energy of the configuration σ_b on b as follows

$$U(\sigma_b) = \frac{1}{2} \sum_{\substack{\langle x,y\rangle, \\ x,y \in V}} \lambda(\sigma(x),\sigma(y)). \tag{3}$$

The following lemma can prove easily.

Lemma 3.1. *For any configuration* φ_b *, we have*

$$U(\varphi_b) \in \{U_{i,n} : i = 0,1,\ldots,k+1, n = 0,1,\ldots,k+1-i\},$$

where

$$U_{i,n} = \frac{i\bar{a} + n\bar{b} + (k+1-i-n)\bar{c}}{2}. \tag{4}$$

Definition 3.1. *A configuration* φ *is called a ground state for the Hamiltonian* H *, if*

$$U(\varphi_b) = min\{U_{i,n} : i = 0,1,\ldots,k+1, n = 0,1,\ldots,k+1-i\}$$

for any $b \in M$.

We denote $C_{i,n} = \{\varphi_b : U(\varphi_b) = U_{i,n}\}$ and

$$A_{\xi,\eta} = \{J \in R^2 : U_{\xi,\eta} = min\{U_{i,n} : i = 0,1,\ldots,k+1, n = 0,1,\ldots,k+1-i\}\}$$

In this condition $k \geq 3$ from (4) and (5) we obtain following:

$$A_{0,0} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{c} \leq \bar{b} \leq \bar{a}\} \cup \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{c} \leq \bar{a} \leq \bar{b}\},$$

$$A_{0,1} = A_{0,2} = A_{0,3} = \ldots = A_{0,k} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{b} = \bar{c} \leq \bar{a}\},$$

$$A_{0,k+1} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{b} \leq \bar{c} \leq \bar{a}\} \cup \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{b} \leq \bar{a} \leq \bar{c}\},$$

$$A_{1,0} = A_{2,0} = \ldots = A_{k,0} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{a} = \bar{c} \leq \bar{b}\},$$

$$A_{1,1} = A_{1,2} = \ldots = A_{1,k-1} = A_{2,1} = \ldots =$$

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$= A_{2,k-2} = \ldots = A_{k-1,1} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{a} = \bar{b} = \bar{c}\}$,

$A_{1,k} = A_{2,k-1} = A_{3,k-2} = \ldots = A_{k,1} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{a} = \bar{b} \le \bar{c}\}$,

$A_{k+1,0} = \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{a} \le \bar{b} \le \bar{c}\} \cup \{(\bar{a},\bar{b},\bar{c}) \in R^3 | \bar{a} \le \bar{c} \le \bar{b}\}$,

and $R^3 = \bigcup_{i,n} A_{i,n}$.

Let $A \subset \{1,2,...,k+1\}$,

$H_A = \{x \in G_k : \sum_{j \in A} w_j(x) - even\}$,

where $w_j(x)$ - is the number of letters a_j in the word x. It is obvious, that H_A is a normal subgroup of index two. Let $G_k/H_A = \{H_A, G_k \setminus H_A\}$ be the quotient group. We set $H_0 = H_A, H_1 = G_k \setminus H_A$.

4 RESULT. PERIODIC GROUND STATES

$H_A -$ periodic configurations has the following form $\sigma(x) = \sigma_i$, if $x \in H_{i-1}$, and $\sigma_i \in \Phi,\ i = 1,2$.

Theorem 4.1. *Let* $k \ge 3$ *and* $|A| = 1$. *If* $|\sigma_1 - \sigma_2| = 0$, *then suitable configurations* σ *in collection* $A_{0,0}$ *are* $H_A -$*periodic ground states; if* $|\sigma_1 - \sigma_2| = 1$, *then suitable configurations* σ *in collection* $A_{0,1}$ *are* $H_A -$*periodic ground states; if* $|\sigma_1 - \sigma_2| = 2$, *then suitable configurations* σ *in collection* $A_{1,0}$ *are* $H_A -$*periodic ground states.*

Proof. Let us consider the following configuration

$$\sigma(x) = \begin{cases} i, & if \quad x \in H_0, \\ i, & if \quad x \in H_1, \end{cases}$$

where $i \in \Phi$. Then $\forall \sigma_b \in C_{0,0}$. Hence, $(\bar{a},\bar{b},\bar{c}) \in A_{0,0}$, so in collection $A_{0,0}$ periodic configuration $\sigma(x)$ is $H_A -$periodic ground state;

Let us consider the following configuration

$$\sigma(x) = \begin{cases} i, & if \quad x \in H_0, \\ j, & if \quad x \in H_1, \end{cases}$$

where $|i - j| = 1,\ i, j \in \Phi$. The folowing cases happen:

Let $c_b \in H_0$. Then $\sigma_b(c_b) = i$, $|B_i| = k$, $|B_j| = 1$, hence $\sigma_b \in C_{0,1}$.

Let $c_b \in H_1$. Then $\sigma_b(c_b) = j$, $|B_i| = 1$, $|B_j| = k$, hence $\sigma_b \in C_{0,1}$.

So in collection $A_{0,1}$ periodic configuration $\sigma(x)$ is $H_A -$periodic ground state;

Let us consider the following configuration

$$\sigma(x) = \begin{cases} i, & if \quad x \in H_0, \\ j, & if \quad x \in H_1, \end{cases}$$

where $|i - j| = 2,\ i, j \in \Phi$. The folowing cases happen:

Let $c_b \in H_0$. Then $\sigma_b(c_b) = i$, $|B_i| = k$, $|B_j| = 1$, hence

$\sigma_b \in C_{1,0}$.

Let $c_b \in H_1$. Then $\sigma_b(c_b) = j$, $|B_i| = 1$, $|B_j| = k$, hence $\sigma_b \in C_{1,0}$.

So in collection $A_{1,0}$ periodic configuration $\sigma(x)$ is $H_A -$periodic ground state.

Theorem has been proved.

5 CONCLUSION

In this paper I considered the $\lambda -$model on the Cayley tree of order $k \geq 3$. Assumed that spin takes its values in the set $\{1,2,3\}$. I calculated energy of all unit balls and colletions assume of minimum value energies of unit balls and I described periodic ground states for the considered model.

ACKNOWLEDGMENT

The author wish to thank Doctor M.M.Rakhmatullaev for formulation of the problem.

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